B.sc(H) part 2 paper 4
Topic:Virtual work
subject Mathematics
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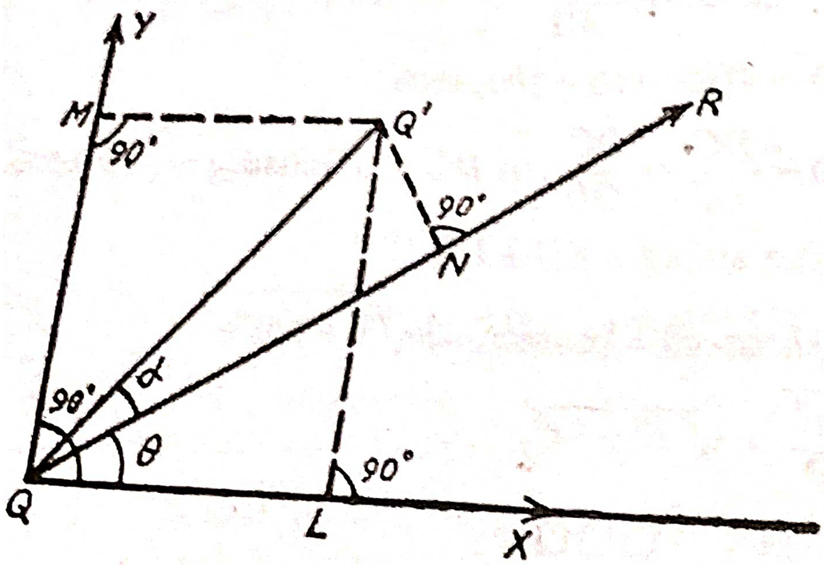
Virtual work

Defination

If there be no motion, no actual displacement is made. However, we may allow the body to receive an imaginary displacement called the virtual displacement. Then the work done by the force in such a virtual displacement is called the virtual work.

To prove that the virtual work done by a force is equal to the sum of the virtual works done by its components.



Proof : Let the point of application of the force R be Q. Let QQ' be the virtual displacement in the plane of the paper and the components of R in two perpendicular directions be X and Y.

If θ be the angle between R and X,

$$X = R\cos\theta,$$

and $Y = R\sin\theta$.

Draw $Q'L$, $Q'M$ and $Q'N$ perpendiculars to X, Y and R respectively. Let $\angle NQQ' = \alpha$.

$\therefore \quad QL = QQ'\cos(\theta+\alpha)$ and $QM = QQ'\sin(\theta+\alpha)$.

Also $\quad QN = QQ'\cos\alpha$.

Then the sum of the virtual works done by X and Y

$$= X.QL + Y.QM$$
$$= R\cos\theta.QQ'\cos(\theta+\alpha) + R\sin\theta.QQ'\sin(\theta+\alpha)$$
$$= R.QQ'.[\cos(\theta+\alpha).\cos\theta + \sin(\theta+\alpha)\sin\theta]$$
$$= R.QQ'\cos(\theta+\alpha-\theta)$$
$$= R.QQ'.\cos\alpha = R.QN$$

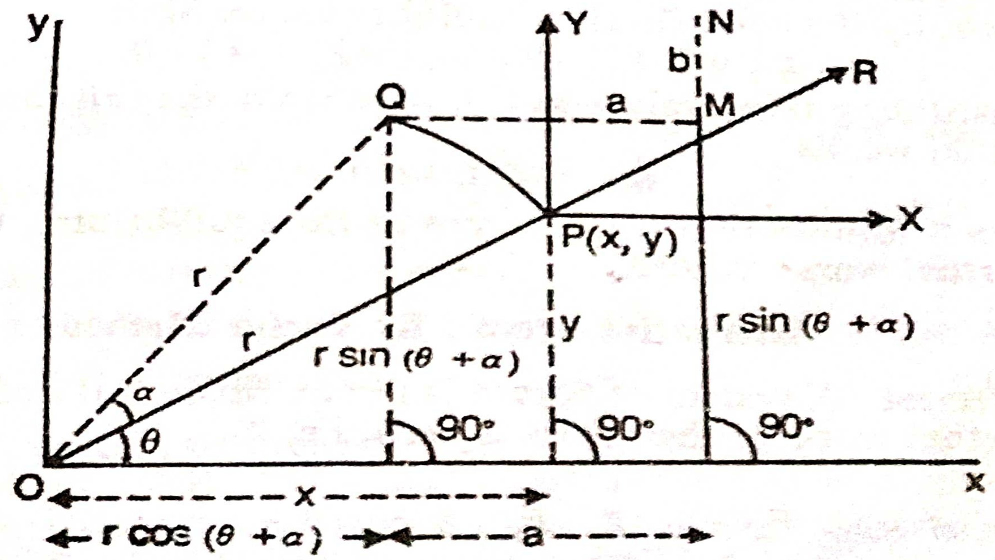
= the vitual work done by R.

.2 Principle of Virtual Work

State and prove the principle of virtual work for any system of forces in one plane.

Answer : *The principle of virtual work states that if a system of forces acting on a body be in equilibrium and the body undergoes a slight displacement consistent with the geometrical conditions of the system, the algebraic sum of the virtual works done by the forces is zero.*

Let Ox and Oy be the two rectangular axes in the plane of the forces. Suppose that any force R of the system acts at the point P



whose cartesian co-ordinates are (x, y) and whose polar co ordinates are (r, θ).

Then $\quad OP = r, \quad \angle xOP = \theta$

and $\quad x = r\cos\theta, \quad y = r\sin\theta.$

Let the body undergo a small displacement. We can do so by rotating the body through a suitable small angle α radians about O and then moving it through suitable distances a and b parallel to the axes.

By the above motion, the point P first moves to Q, then to M and finally to N.

The co-ordinates of N are

$$\{r\cos(\theta+\alpha)+a,\ r\sin(\theta+\alpha)+b\}$$

i.e. $\{r(\cos\theta\cos\alpha-\sin\theta\sin\alpha)+a,\ r(\sin\theta\cos\alpha+\cos\theta\sin\alpha)+b\}$

i.e. $\{r(\cos\theta.1-\sin\theta.\alpha)+a,\quad r(\sin\theta.1+\cos\theta\ \alpha)+b\}$, neglecting square and higher powers of the small angle α

i.e. $(r\cos\theta-\alpha\ r\sin\theta+a,\ r\sin\theta+\alpha.r\cos\theta+b)$

i.e. $(x-\alpha y+a,\ y+\alpha x+b)$.

The changes in the co-ordinates of N are therefore

$$x-\alpha y+a-x \text{ and } y+\alpha x+b-y$$

i.e. $a-\alpha y$ and $b+\alpha x$.

Let X and Y be the components of the force R parallel to the co-ordinate axes.

Then virtual work of R = virtual work of X + virtual work of Y

$$=X(a-\alpha y)+Y(b+\alpha x)$$

$$=aX+bY+\alpha(xY-yX).$$

Similar expressions for the virtual works done by the other forces can be obtained; a, b and α being the same for each force.

The algebraic sum of the virtual works is therefore

$$=a\Sigma X+b\Sigma Y+\alpha\Sigma(xY-yX). \qquad \text{.. (1)}$$

Since the system of forces acting on the body is in equilibrium, therefore, by the conditions of equilibrium, we have

$$\Sigma X=0,\ \Sigma Y=0 \text{ and } \Sigma(xY-yX)=0.$$

Substituting these values in (1), we obtain the algebraic sum of the virtual works

$$=a.0+b.0+\alpha.0=0.$$

Thus it follows that if the forces be in equilibrium, the sum of their virtual works is zero.

Alternative Proof : By Vector Method

Statement. A system of forces is in equilibrium if and only if the total virtual work of the forces is zero, i.e. if

$$\sum_{v=1}^{N} \vec{F}_v \cdot \delta \vec{r}_v=0.$$

This is the principle of virtual work.

Proof. In order for a system of forces to be in equilibrium, the resultant force acting on each particle must be zero,

i.e. $$\vec{F}_v=\vec{0}.$$

Hence we get $$\vec{F}_\nu \cdot \delta \vec{r}_\nu = 0,$$
where $\vec{F}_\nu \cdot \delta \vec{r}_\nu$ is the virtual work.

By adding these, we get

$$\sum_{\nu=1}^{N} \vec{F}_\nu \cdot \delta \vec{r}_\nu = 0.$$

.3 Converse of the Principle of Virtual Work

State and prove the converse of the principle of virtual work for any system of forces in one plane.

Statement · *If the algebraic sum of the virtual works done by the system of forces in any small displacement consistent with the geometrical conditions of the system is zero, the forces are in equilibrium.*

Answer : Deduce the result (1) of Art. 2.2

i.e. the algebraic sum of the virtual works
$$= a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX)$$
and this is given to be zero for all displacements.
$$\therefore \quad a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX) = 0. \quad \text{.. (2)}$$
We may give any values to a, b, α as they are arbitrary quantities. Let $b=0$, $\alpha=0$ but $a \neq 0$.

This means that the displacement is to be given to the body through a distance a parallel to the x-axis only.

So, from (2), $a\Sigma X + 0\Sigma Y + 0\Sigma(xY - yX) = 0$; i.e., $a\Sigma X = 0$;
$$\therefore \quad \Sigma X = 0, \text{ as } a \neq 0.$$
Similarly choosing $a=0$, $\alpha=0$ but $b \neq 0$, we obtain from (2),
$$b\Sigma Y = 0.$$
$$\therefore \quad \Sigma Y = 0 \text{ as } b \neq 0.$$
Finally, let the displacement be one of simple rotation about O.

So, $$a=0,\ b=0 \text{ and } \alpha \neq 0.$$
$\therefore$ From (2), $$\alpha\Sigma(xY - yX) = 0$$
$$\therefore \quad \Sigma(xY - yX) = 0, \text{ as } \alpha \neq 0.$$
Thus we get $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma(xY - yX) = 0$.

But these are the conditions of equilibrium. Hence the system of forces is in equilibrium.